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Properties of inviscid, recirculating flows

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Integral relations are derived for steady, incompressible recirculating motions with small viscous forces. The circuit time of a fluid particle on a closed streamline in steady, inviscid flow is shown to be the same for all the closed streamlines on a surface of constant total head.

The discontinuities of velocity and velocity gradient that occur in the motion of inviscid fluid filling a closed, rotating cylinder set in a rotating support with the two rotation axes slightly misaligned are then investigated.

1. Introduction

Recirculating flows occur in wakes behind bluff bodies and in cavities in windswept surfaces. A significant advance in their theoretical treatment was the discovery of simple rules for the distribution of vorticity in steady, incompressible, two-dimensional or axisymmetric flows with small viscous forces. The arguments leading to these rules, however, require the symmetries to be accurate to $o(R^{-1})$ for large Reynolds number R. This requirement is exacting and makes flows which depart from these special symmetries, even slightly, worth considering.

In the first half of this paper a few general properties of steady, recirculating flows are derived from integrals of the equations of motion. Foremost are the field equations (2.3) and (2.4) which supplement Euler's equations of inviscid motion. These apply when, in the inviscid limit, there is no normal component of vorticity at the enveloping boundaries. In particular they confirm the simple rules for closed axially symmetric flows with velocity components in planes containing the axis as an appropriate approximation when small asymmetries are present. The asymmetries here may be independent of Reynolds number. Viscous forces though small are again crucial and extraneous forces, notably buoyancy, are required to be smaller still. For certain flows where the normal vorticity at the boundary is not zero in the inviscid limit, an extension is noted of the mechanism proposed by Batchelor (1951) for the control of the inviscid core between two rotating disks. The curious property is also noted that the circuit time for particles on closed streamlines in steady, strictly inviscid flow is the same for any two closed streamlines on a surface of constant total head H, provided that on the surface the streamlines are reconcilable and $\nabla H \neq 0$. All these properties extend to flows steady relative to a rotating frame.

The uniform vorticity predicted for two-dimensional flows with small viscous forces is not necessarily a good approximation when the flow is slightly threedimensional. The axisymmetric flow between two rotating spheres (Proudman 1956) can be two-dimensional in the inviscid limit and can at the same time have a non-uniform vorticity distribution, because the axial drift of $O(R^{-\frac{1}{2}})$ from the sphere's boundary layers has a larger effect than two-dimensional viscous diffusion. End effects can also be important in other ways. The experiments of Maull & East (1963) on wind-swept cavities with a span several times longer than the depth and the streamwise-width indicate a regular spanwise arrangement of boundary-layer separations. While a rigidly rotating column is well known to possess inviscid, asymmetric modes of disturbance which can be resonantly amplified at large Reynolds numbers by a column of suitable length (cf. Chandrasekhar 1961). Inertial resonances of this kind were described by Kelvin (1880), for oscillatory disturbances, and were pursued by Bjerknes & Solberg (1933) because of their possible bearing on large-scale atmospheric circulations. They are of interest too because the governing linearized equations are hyperbolic in planes containing the axis of rigid rotation. Some novel consequences of this special aspect of small departures from two-dimensional flow are discussed in the second half of the paper.

The motion will be considered of inviscid fluid filling a finite cylinder which rotates about its axis and is set at a small angle to the vertical in a frame which rotates about the vertical. The flow is steady relative to the cylinder and is subject to resonance. The additional feature described here is the pattern of discontinuities of velocity and shear that can occur in the fluid interior. The pattern moreover changes randomly with changes in the ratios, cell height/cell radius and precessional angular speed/primary angular speed.

Perturbations, steady relative to a rotating frame, of axisymmetric Couette flows have a hyperbolic, linearized equation for the component of disturbance pressure which varies as $e^{in\psi}$ if

$$2v_0\, d(\rho v_0)/d\rho > n^2(v_0 - \rho\omega_2)^2.$$

Here $\omega_2 \mathbf{k}$ is the frame's angular velocity, ρ, ψ are polars in planes normal to $\mathbf{k}$ with ψ measured relative to the rotating frame and $v_0(\rho)$ is the unperturbed speed relative to fixed axes. It seems probable that the same qualities of discontinuity and randomness are shared by all such disturbances. For steady disturbances of rigid rotation the first asymmetric mode is included but none of the higher modes.

2. General properties

We shall derive certain integrals for regions of small viscous forces in steady, recirculating, incompressible motions. In the absence of special symmetries, it is convenient to specify the topology of the surfaces of constant total head H. We distinguish (a) flows, like a ring vortex, where the H-surfaces lie wholly in the inviscid zone and are closed, and (b) flows, as between rotating disks, where the H-surfaces pass into shear layers (figure 1). ∇H will be assumed non-zero.

Suppose first that the total head is constant on closed surfaces lying wholly in the interior of the inviscid zone. The equations of motion, written

$$\mathbf{u} \times \boldsymbol{\omega} = \nabla H + \nu \operatorname{curl} \boldsymbol{\omega}, \quad \operatorname{div} \mathbf{u} = 0, \tag{2.1}$$

where $\mathbf{u}$ is the velocity and $\boldsymbol{\omega}$ the vorticity, give for the volume ΔV between any two, closed total head surfaces $H = H_0$, $H = H_0 + \Delta H$

$$\int_{\Delta V} \mathbf{u} \,.\, \text{curl}\, \boldsymbol{\omega}\, d\tau = 0, \tag{2.2}$$

provided there are no sources or sinks in the volume enclosed by either surface. Thence on dividing by ΔH and letting $\Delta H \to 0$, we have for each surface $H = H_0$,

$$\int_{H=H_0} \frac{\mathbf{u} \,.\, \text{curl}\, \boldsymbol{\omega}}{|\nabla H|}\, dS = 0. \qquad (2.3$$

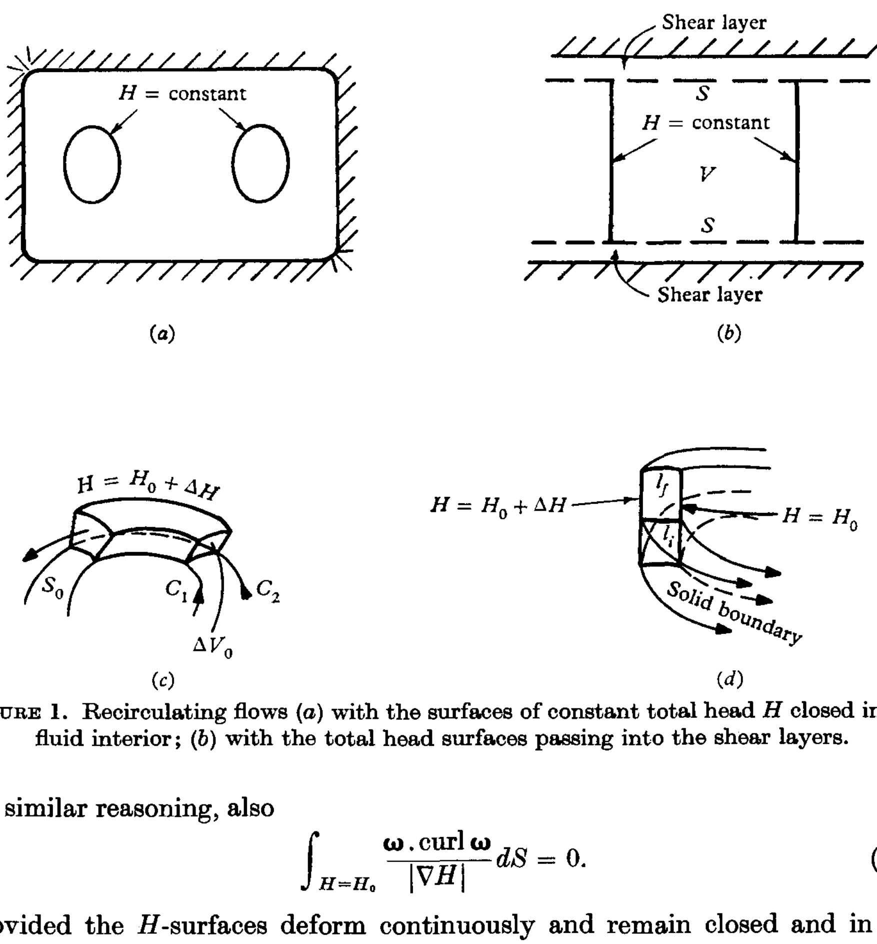

FIGURE 1. Recirculating flows (*a*) with the surfaces of constant total head H closed in the fluid interior; (*b*) with the total head surfaces passing into the shear layers.

By similar reasoning, also

$$\int_{H=H_0} \frac{\boldsymbol{\omega} \,.\, \text{curl}\, \boldsymbol{\omega}}{|\nabla H|}\, dS = 0. \tag{2.4}$$

Provided the H-surfaces deform continuously and remain closed and in the inviscid zone as $\nu \to 0$, all the integrals hold also in the inviscid limit.

The integrals thus yield in effect field equations supplementary to the Euler equations of inviscid motion, and though simply derived are not without significance. For axisymmetric flows, (2.3) and (2.4) yield the rules for vorticity and azimuthal velocity found by Batchelor (1956) (cf. his equations (4.14) and (4.17)), with the minor addition that these rules can now be extended to motions with an internal boundary. Where the vortex lines are co-ordinate lines of an orthogonal system, as in most simple geometries, (2.4) is satisfied automatically.

We now pass to flows where the total head surfaces of the inviscid zone pass into shear layers (figure 1 (*b*)). For a volume V in the inviscid zone enclosed by a total head surface $H = H_0$ and an aggregate surface S, the equations of motion give

$$-\int_S \mathbf{u}H\,.\,d\mathbf{S} = \nu\int_V \mathbf{u}\,.\,\mathrm{curl}\,\boldsymbol{\omega}\,d\tau. \tag{2.5}$$

The surface S may be taken close to the shear layers relative to the inviscid zone's overall dimensions. When S is close enough, the velocity u_n normal to S is of the order of the efflux velocity at the outer edge of the shear layer. Hence for small ν the left-hand side of (2.5) predominates, and to a first boundary-layer approximation,

$$\int_S u_n H\,dS = 0, \quad \int_C \frac{u_n \sin\theta}{|\nabla H|}ds = 0. \tag{2.6}$$

In the last integral, θ is the angle between ∇H and the normal to S, and C is the aggregate boundary of S, i.e. C is the aggregate of the close loops defined on the surface $H = H_0$, of the inviscid zone, at the outer 'edge' of each shear layer which it intercepts.

Control of the inviscid core by the requirement that the efflux and influx of opposing boundary layers should match was the mechanism advanced for rotating disks (Batchelor 1951). The burden of (2.6) is that the inviscid core is controlled similarly *whenever* the equi-total head surfaces pass into the bounding shear layers.

Flows with small viscous forces which differ slightly remain in the same group (*a*) or (*b*), as defined above, provided that $\nabla H \neq 0$. So the integrals obtained apply to slightly non-axisymmetric flows. In particular they confirm that (*a*) the known vorticity rules for axisymmetric flows with helical streamlines and (*b*) the boundary-layer matching mechanism for axisymmetric flows with azimuthal streamlines and boundaries oblique to the axis are both good approximations in the presence of small asymmetries. The asymmetries may be $O(1)$ for large Reynolds numbers. The uniform vorticity rule, by contrast, is not necessarily a good approximation in the presence of small departures from two-dimensionality because, as remarked in the introduction, convections of $O(R^{-\frac{1}{2}})$ can dominate the effects of two-dimensional diffusion.

A further property of steady (exactly) inviscid flow concerns streamlines which are closed. From the inviscid equations, i.e. (2.1) with $\nu = 0$, we can write

$$\boldsymbol{\omega} = \lambda\mathbf{u} - (\mathbf{u}\times\nabla H)/q^2, \tag{2.7}$$

where λ is a scalar and $q = |\mathbf{u}|$. As remarked at the outset ∇H is presumed non-zero, and stagnation points are consequently excluded. On taking the divergence of (2.7), there follows

$$0 = \mathrm{div}\,\lambda\mathbf{u} - (\mathrm{curl}\,\mathbf{u}/q^2)\,.\,\nabla H. \tag{2.8}$$

We now consider any pair of closed and reconcilable streamlines C_1 and C_2 on a surface $H = H_0$. The two closed streamlines enclose an area S_0 on the surface $H = H_0$, and the normals to S_0 form a small volume ΔV_0 between $H = H_0$ and a neighbouring surface $H = H_0 + \Delta H$ (figure 1 (*c*)). Fluid enters the volume ΔV_0 only through a strip of width $O(\Delta H)$ and at a small angle to the strip. Hence the inward flux is $o(\Delta H)$ for small ΔH. Integration of (2.8) over the volume ΔV_0 therefore gives

$$0 = \lim_{\Delta H \to 0} \frac{1}{\Delta H} \int_{\text{strip}} \mathbf{u}\lambda . d\mathbf{S}$$

$$= \lim_{\Delta H \to 0} \frac{1}{\Delta H} \int_{\Delta V_0} \operatorname{curl} \frac{\mathbf{u}}{q^2} . \nabla H \, d\tau$$

$$= \int_{S_0} \operatorname{curl} \frac{\mathbf{u}}{q^2} . d\mathbf{S}; \tag{2.9}$$

whence, finally,

$$\int_{C_1} \frac{\mathbf{u}}{q^2} . d\mathbf{S} = \int_{C_2} \frac{\mathbf{u}}{q^2} . d\mathbf{S}. \tag{2.10}$$

We conclude that, in steady, inviscid, incompressible flow, the circuit times of fluid particles on any pair of closed streamlines on a surface of constant total head are equal, provided that on the surface the streamlines can be deformed continuously into each other and $\nabla H \neq 0$.*

The existence of closed streamlines in inviscid flow is again linked with the topology of the total head surfaces. The simplest case is where the H-surfaces end on solid boundaries and are topologically like the H-surfaces in rigid rotation between two rotating disks. The streamlines are then all closed. To see this, we may picture a slender stream-tube bounded by two H-surfaces at the sides, by the solid boundary at the base and by a strip of streamlines on top (figure 1 (*d*)). If the 'top' strip of streamlines does not close it has, after circulating round the boundary, an intercept l_f with any plane roughly normal to the tube different from its initial intercept l_i. The line intercepts l_i and l_f must intersect, because the same total flux passes through each of the initial and final cross-sections in the plane (i.e. one cross-section cannot be inside the other). At least one streamline leaving l_i therefore circulates back to l_i. This is incompatible with any streamline near the boundary being unclosed, since l_i is arbitrary and can be indefinitely short. All the streamlines near the boundary are therefore closed—and so on for the whole region. The circuit time, moreover, varies continuously with H. The circuit time result applies, for example, to axisymmetric flows, with azimuthal streamlines and rigid boundaries normal to the axis, which are asymmetrically perturbed.

The closure of streamlines cannot be assumed if the H-surfaces are closed and do not intersect the solid boundaries. This is illustrated by the axisymmetric ring vortex with azimuthal swirl. The helical streamlines on each toroidal H-surface are all closed or all unclosed depending whether the ratio of the times for single circuits of the straight axis and the central, circular axis is rational or irrational. The ratio normally changes continuously with H and the accompanying change in closure is highly erratic. The value of the circuit-time result for closed surfaces is, correspondingly, restricted.

All the above arguments have simple counterparts for motions which are steady relative to a frame rotating with constant angular velocity $\boldsymbol{\omega}_2$. The equations of motion are then

$$\mathbf{u} \times (\boldsymbol{\omega} + 2\boldsymbol{\omega}_2) = \nabla H + \nu \operatorname{curl} \boldsymbol{\omega}, \tag{2.11}$$

* This was first proved by Ertel (1950) by a different method for flows in which all the streamlines are closed (cf. Truesdell 1954).

where $\mathbf{u}$ and $\boldsymbol{\omega}$ are now relative to the frame and H now includes $-\boldsymbol{\omega}_2^2 \times$ (distance to the axis)^2. With these reinterpretations of $\mathbf{u}$, $\boldsymbol{\omega}$ and H, the integral conditions, (2.3), (2.4) and (2.6), on the inviscid flow apply as before, except that (2.4) now reads

$$\int_{H=H_0} \frac{\boldsymbol{\omega}+2\boldsymbol{\omega}_2}{|\nabla H|} . \operatorname{curl} \boldsymbol{\omega} \, dS = 0. \tag{2.12}$$

The circuit-time result also holds for the relative velocity, because div $(\boldsymbol{\omega}+2\boldsymbol{\omega}_2)$ still vanishes, and the remarks on streamline closure carry over unchanged.

3. A perturbed inviscid rigid rotation

The next two sections are devoted to a pattern of internal discontinuities of velocity and shear that occurs when a rigid rotation is disturbed. The perturbation in question is steady relative to a rotating frame. However, as mentioned

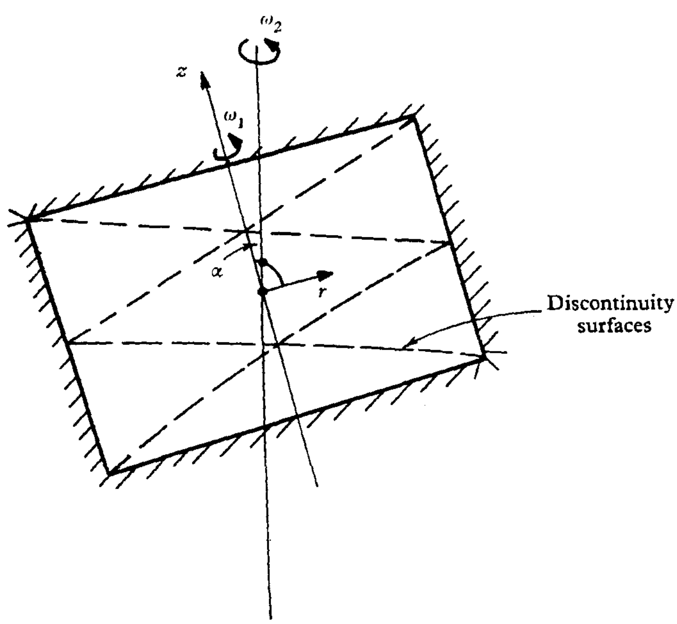

FIGURE 2. Rotating fluid cell with discontinuity surfaces.

above, the same features are expected to be shared by the first asymmetric mode of a steady, inviscid disturbance of rigid rotation, because the linearized equation for the relevant pressure disturbance is again hyperbolic.

The motion to be considered is of inviscid fluid filling a closed cylinder which is set in a frame at a small angle α to the vertical and rotates about its own axis with angular velocity $\boldsymbol{\omega}_1$, relative to the frame, while the frame rotates with angular velocity $\boldsymbol{\omega}_2$ about the vertical (figure 2). The cylinder centre is supposed at rest.

Relative to the rotating support the motion is steady. Polars (r, ϕ, z) fixed in the support will be used, with $\mathbf{z}$ along the cylinder's axis and $\mathbf{r}$ in the plane of the two axes of rotation. After writing the velocity relative to the support as

$$\mathbf{u} = r\omega_1 \Phi + \alpha(u\mathbf{r} + v\Phi + w\mathbf{z}) \tag{3.1}$$

and ignoring second-order quantities in α, the equations of motion become

$$\partial u/\partial\phi - 2\omega v = -\partial\Pi/\partial r, \quad \partial v/\partial\phi + 2\omega u = -\partial\Pi/r\,\partial\phi,$$
$$\partial w/\partial\phi + 2r\omega_2\cos\phi = -\partial\Pi/\partial z, \quad \partial(ru)/\partial r + \partial v/\partial\phi + r\,\partial w/\partial z = 0, \tag{3.2}$$

where Π is a reduced pressure and $\omega = 1+\omega_2/\omega_1$. Thence with

$$u = \omega_1 U\sin\phi, \quad v = \omega_1 V\cos\alpha, \quad w = \omega_1 W\sin\phi, \quad \Pi = \omega_1 Q\cos\phi, \tag{3.3}$$

we have

$$U = \left(\frac{\partial Q}{\partial r} + \frac{2\omega Q}{r}\right)\Big/(4\omega^2-1), \quad V = \left(2\omega\frac{\partial Q}{\partial r} + \frac{Q}{r}\right)\Big/(4\omega^2-1),$$
$$W = -\partial Q/\partial z + 2r(1-\omega) \tag{3.4}$$

and
$$\frac{\partial^2 Q}{\partial r^2} + \frac{1}{r}\frac{\partial Q}{\partial r} - \frac{Q}{r^2} - (4\omega^2-1)\frac{\partial^2 Q}{\partial z^2} = 0. \tag{3.5}$$

A formal solution satisfying the boundary conditions is then, for $2\omega > -1$,

$$Q = 4(\omega-1)(2\omega+1)(4\omega^2-1)^{\frac{1}{2}}a^2\sum_{s=1}^{\infty}\frac{J_1(\lambda_s r/a)\sin\{\lambda_s z/a(4\omega^2-1)^{\frac{1}{2}}\}}{\lambda_s(\lambda_s^2+4\omega^2-1)J_1(\lambda_s)\cos\{\lambda_s l/a(4\omega^2-1)^{\frac{1}{2}}\}} \tag{3.6}$$

where a and l are respectively the cylinder's radius and half-height and the λ_s are the positive roots of $\lambda J_1'(\lambda)+2\omega J_1(\lambda) = 0$. If $2\omega < -1$, there are two imaginary roots of the eigen-equation for λ and two extra terms have to be inserted in the series for Q. The motion is well behaved if $|2\omega| < 1$: the interest arises when $|2\omega| > 1$ and the pressure equation (3.5) is hyperbolic. As observed by Kelvin (cf. Chandrasekhar 1961) resonance occurs whenever

$$l/\pi a(4\omega^2-1)^{\frac{1}{2}}\,(= c/\pi, \text{say}) = (\text{odd integer})/2\lambda_s \tag{3.7}$$

To discuss the singular surfaces involved in the above solution, attention is directed to rational values of c (namely R/T where R and T are co-prime). These values are the ones for which the characteristics,

$$r/a \pm cz/l = \text{const.} \tag{3.8}$$

are closed after one or more reflexions at the cylinder's sides, ends or axis.

The resonant $c = (\text{odd integer})\pi/2\lambda_s$ and the rational c selected are both denumerable, dense sets in $(0,\infty)$. It is *a priori* unlikely that a given rational should be of the form $(\text{odd integer})\pi/2\lambda_s$ and the possibility of resonant infinities for the selected c will be ignored in what follows. However, it should be mentioned that so doing strictly involves an assumption.

4. Velocity and shear discontinuities

(*a*) $R \neq 2(\text{mod}\,4)$

We note that when s is large

$$\lambda_s = (s-s_0+\tfrac{3}{4})\pi + (2\omega-\tfrac{7}{8})/\pi s + O(1/s^2), \tag{4.1}$$

$\cos\{\lambda_s l/a(4\omega^2-1)^{\frac{1}{2}}\}$

$$= (-1)^{\zeta R}[\cos\{(\sigma+\tfrac{3}{4})R\pi/T\} - (2\omega-\tfrac{7}{8})R\sin\{(\sigma+\tfrac{3}{4})R\pi/T\}/\pi Ts + O(1/s^2)],$$

where s_0 is a fixed integer and $s-s_0$ is expressed $\zeta T+\sigma$ with $\sigma = 0, 1, 2, \ldots, T-1$. The pressure modes are thus $O(1/s^3)$ provided $R \neq 2 (\text{mod } 4)$ and on collecting the leading approximations to the higher-order modes, the pressure may be written,

$$Q = af\left(\frac{r}{a}+\frac{Rz}{Tl}\right) - af\left(\frac{r}{a}-\frac{Rz}{Tl}\right)$$

$$+ \text{contributions which yield continuous velocity gradients,} \tag{4.2}$$

where

$$\left.\begin{aligned} f(\alpha) &= \frac{2(\omega-1)(1+2\omega)}{\pi}\left(\frac{(4\omega^2-1)a}{r}\right)^{\frac{1}{2}} \operatorname{Re}[kF], \\ k(\alpha) &= \frac{1}{T}\sum_{\sigma=0}^{T-1} \frac{\exp\{\pi i[(\sigma+\frac{3}{4})\alpha-\frac{1}{4}]\}}{(-1)^{\sigma}\cos(\sigma+\frac{3}{4})R\pi/T}, \\ F(\alpha) &= -\int_{\alpha_0}^{\alpha}\int_{\alpha_0}^{\alpha} \log\{1-\exp[\pi i(\alpha' T+R+T)]\}\, d\alpha'\, d\alpha. \end{aligned}\right\} \tag{4.3}$$

α_0 is such that the logarithm is bounded.

The velocity gradients are evidently liable to discontinuities when

$$rT/a \pm zR/l + R + T = \text{an even integer}, \tag{4.4}$$

because of the singular behaviour of $d^2F/d\alpha^2$. The singular surfaces are readily identified as the cones generated by the characteristics through the corners $r = a$, $z = \pm l$ and their reflexions at the cylinder's ends, sides, and axis. The discontinuity occurs in the gradient normal to the singular surface of the component of velocity in a direction $\boldsymbol{\tau}$ lying in the surface and at angle ϕ (taken positive in the sense r increasing) to Φ. The other components of shear for orthogonal axes based on these two directions are continuous except on the axis and on the circles where the singular characteristics cross.

The shear wave strength varies in the axial plane as $r^{-\frac{1}{2}}$. The coefficient k on a corner characteristic $r/a+Rz/Tl = \alpha+$ or $r/a-Rz/Tl = \alpha-$ can be shown after some manipulation to be

$$k = 2(-1)^m i^{\frac{1}{2}(3\mu-1)}/(1-i^{2T-R}) \tag{4.5}$$

where μ is the single, odd integral value of

$$(\alpha\pm)+(2m+1)R/T \quad \text{for} \quad m = 0, 1, \ldots (T-1).$$

Thus $|\operatorname{Re} k| = |\operatorname{Im} k| = 1$ and the shear-wave strength varies smoothly with ω and l/a. The pattern of discontinuities, however, varies erratically with ω and l/a, because a very small change in the semi-angle of the characteristic cones can mean a very large change in the number of reflexions of the corner characteristics at the cylinder sides, ends and axis. It will be recalled that the characteristics from a corner ultimately reflect back to a corner because of the choice of rational values of $c = l/a(4\omega^2-1)^{\frac{1}{2}}$.

The nature of the discontinuity depends on k as shown in table 1. The discontinuity when $R = 0 (\text{mod } 4)$ is finite on the characteristics at the corners. Reflexion at the ends and sides leaves the type of discontinuity unchanged, the sign of the operative part of k being such as to keep $\partial^2\Pi/\partial r\,\partial z$ continuous along the boundary. Reflexion at the axis, however, is not so simple. For the $\alpha+$ and $\alpha-$ characteristics of a reflexion pair at the axis,

$$(\alpha+)+(\alpha-) = 0, \tag{4.6}$$

and we have in turn

$$\begin{aligned} [(m+)+(m-)+1]\,2R/T &= (\mu+)+(\mu-), \\ m+ &= T-1-(m-), \\ \mu+ &= 2R-(\mu-), \\ k+ &= -i/(k-) \quad \text{for} \quad R = 0(\text{mod}\,4). \end{aligned} \tag{4.7}$$

Consequently when $R = 0(\text{mod}\,4)$ a finite discontinuity reflects at the axis as a logarithmically infinite discontinuity and vice versa. A similar, more protracted argument shows that in the second case $R = 1$ or $3(\text{mod}\,4)$ both of the two possibilities $\text{Re}\,(k+) = \pm\,\text{Re}(k-)$ occur at different reflexion points (if there are more than one) along the axis.

$R = 0(\text{mod } 4)$	$R = 1$ or $3(\text{mod } 4)$
$k = (-1)^m i^{(3\mu-1)/2}$	$\text{Re}\,k \neq 0$
Discontinuity alternatively finite and logarithmically infinite on neighbouring parallel 'corner' characteristics	Discontinuity logarithmically infinite on all 'corner' characteristics

TABLE 1.

(*b*) $R = 2(\text{mod}\,4)$

Pressure modes of $O(1/s^2)$ now recur for $s-s_0 = \zeta T+\sigma^*$ where

$$\begin{aligned} 4\sigma^*+3 &= T \quad \text{if} \quad T = 3(\text{mod}\,4) \\ &= 3T \quad \text{if} \quad T = 1(\text{mod}\,4) \end{aligned} \tag{4.8}$$

and the pressure may now be written

$$Q = ag\left(\frac{r}{a}+\frac{Rz}{Tl}\right) - ag\left(\frac{r}{a}-\frac{Rz}{Tl}\right)$$

$$+\text{ contributions which yield continuous velocities,} \tag{4.9}$$

where

$$g(\alpha) = \frac{16(\omega-1)(1+2\omega)}{\pi(16\omega-7)R}\left(\frac{(4\omega^2-1)a}{r}\right)^{\frac{1}{2}} \text{Re}\left\{\frac{\exp\left[\frac{1}{4}\pi i((1 \text{ or } 3)\alpha T+1)\right]}{(-1)^{[(1 \text{ or } 3)(R+T)-1]/4}}\frac{dF}{d\alpha}\right\} \tag{4.10}$$

and (1 or 3) is taken as 1 if $T = 3(\text{mod}\,4)$ and vice versa.

Thus the velocity in the tangential direction $\boldsymbol{\tau}$ is now itself discontinuous. The strength of the discontinuity varies in an axial plane as $r^{-\frac{1}{2}}$ and now varies also inversely as the number of reflexions R of the original corner characteristic by the axis. An examination of the parity of $((1 \text{ or } 3)\,\alpha T+1)/2$ reveals that the type of discontinuity changes between finite and logarithmically infinite on reflexion at the axis. There is no change in type on reflexion at the sides or ends, because the normal velocities there are zero.

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